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# SELF-SIMILAR SOLUTIONS OF A SYSTEM OF TWO PARABOLIC EQUATIONS

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The description of many physical systems comes down to the solution of a system of two nonlinear equations of the parabolic type. Such systems can be the electron–hole plasma of a semiconductor and a weakly ionized gas plasma, nonequilibrium superconductors, as well as a number of chemical and biological objects, the properties of which are determined by autocatalytic reactions. The formation of complicated nonuniform structures occurs upon the loss of stability in these systems. We shall examine the concrete problem of the development of an ionization–superheating instability in a self-maintained discharge, described by the equation of charged-particle balance of the plasma and the equation of heat balance. The mechanism of this stability is connected with the decrease in the density of gas escaping at constant pressure from a superheated region, and with the rise in electron temperature occurring as a consequence of this (see, e.g., [1]). Self-similar functions for the local values of the charged-particle density and the gas temperature, being solutions of the corresponding balance equations, are of interest for the understanding of the nonlinear state of this process. An ionization–superheating instability in a high-frequency field and a self-similar solution, describing the explosive development of conductivity in a constricting discharge, neglecting the thermal conductivity of the gas and charge recombination, were studied in [2]. Self-similar solutions of a pair of equations of the parabolic type under the conditions of a self-maintained glow discharge are investigated in the present paper. The solutions obtained can be of interest for a whole series of physical systems.

Let an electric discharge be ignited between two electrodes spaced a distance L apart. Assuming that it is uniform along the current, we use balance equations for the charged-particle density n and the gas temperature T:

$$\partial n/\partial t - D_a \Delta n = \nu_i n - \beta n^2; \tag{1}$$

$$\frac{1}{T}\frac{\partial T}{\partial t} - \chi\frac{1}{T}\Delta T = \frac{\sigma E^2}{c_p p}. \tag{2}$$

Here $D_a$ and $\chi$ are the coefficients of ambipolar diffusion and thermal diffusivity, respectively; $\nu_i$ is the frequency of ionization by electron impact; $\beta$ is the dissociative-recombination constant; $c_p$ is the reduced heat capacity of the gas; $\sigma = e^2 n/m\nu_m$ is the conductivity of the discharge plasma, which neglecting electron–electron collisions, is proportional to the electron density. In writing (1) and (2), it was assumed that the time of pressure equalization is small compared with the characteristic time of development of instability. This is possible if the pressure does not increase with time owing to the presence of a large ballast volume.

The ionization frequency is usually a sharply growing function of the parameter $E/N \sim ET$ (N is the gas density). Under the conditions of a gas discharge, the approximation $\nu_i = A \exp(-Bp/ET)$ is used for the frequencies [1], where A, B = const.

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For relatively small variation of the quantities $\Delta E/E_0$ and $\Delta T/T_0$, one can adopt an expansion of this expression,

$$\nu_i = \nu_{i0}\exp[C(\Delta T/T_0 + \Delta E/E_0)], \tag{3}$$

with $C = Bp/E_0T_0 \gg 1$. The remaining coefficients depend little on the field and temperature, so that their variation is neglected.

With an assigned external source having an emf $\varepsilon$ and an electric field strength $E_0 = \varepsilon/L$, for the dimensionless quantities $u = \Delta T/T_0$ and $v = n/n_0$ we have the system of equations ($n_0$ and $T_0$ are the values of the electron density and gas temperature)

$$\partial v/\partial t - D_a\Delta v = \nu_{i0}v\exp(Cu) - \beta v^2; \tag{4}$$

$$\partial u/\partial t - \chi\Delta u = \nu_n v \tag{5}$$

($\nu_n^{-1} = c_p p/\sigma(n_0)E_0^2$ is the heating time of the neutral gas).

If the walls confining the discharge in the transverse direction are located far enough away that heat removal in them can be neglected, then the uniform solution of the system (4), (5) describes quantities u and v growing slowly with time. With weak feedback through an external circuit, far faster explosive growth of nonuniform distrubances is possible against this background. A mathematically similar situation arises in the propagation of heat in media containing distributed nonlinear sources [3-5]. Here it turns out that the initial disturbances can be concentrated into a filament where the temperature becomes as high as desired in a finite time.

The development of such disturbances obeys certain self-similar relations. We seek the solution of the system (4), (5) in the form

$$u = \frac{\psi(\tau) - \ln[(t_0 - t)\nu_{i0}]}{C}; \tag{6}$$

$$v = \frac{\varphi(\tau)}{C\nu_n(t_0 - t)}, \tag{7}$$

where $t_0$ is the time of the explosion; $\tau = x/2\sqrt{D_a(t_0 - t)}$ is the self-similar variable. Here and later we consider the one-dimensional case, when all the quantities vary along the transverse current of the spatial variable x. All the results also remain valid in a cylindrical geometry with the axis coinciding with the position of the generatrix of the current filament.

The functions $\phi$ and $\psi$ satisfy a system of two ordinary differential equations:

$$\varphi + \frac{\tau}{2}\varphi' - \frac{\varphi''}{4} = \varphi e^{\psi} - k\varphi^2; \tag{8}$$

$$1 + \frac{\tau}{2}\psi' - \varkappa\frac{\psi''}{4} = \varphi. \tag{9}$$

The system contains the parameter $\kappa = \chi/D_a$ and $k = \beta n_0/\nu_n C$, the magnitude of which determines the relative roles of heat conduction and volumetric recombination. In a glow discharge the first of these is small. Taking $\chi$ as approximately equal to the coefficient of molecular diffusion, we obtain $\varkappa \simeq T/T_e \simeq 3\cdot10^{-2}$. Under the assumptions made above, the parameter k does not depend on $n_0$ and, for a concrete type of gas, is a function of the argument E/N. Estimates show that it can be either greater or less than one. For nitrogen with $E/N = 6\cdot10^{-16}$ $V\cdot cm^2$, e.g., we have $\kappa \cong 10$, while for helium with $E/N = 10^{-16}$ $V\cdot cm^2$ we have $k \cong 0.1$.

Let us consider certain peculiariies of solutions of the system (8), (9) that have physical meaning for any value of $\tau$. The equations take the simplest form for $\kappa = k = 0$. In this case, their order can be reduced to one, for which we multiply the first of the equations by $\tau$ and find its first integral, substituting the value of $\phi$ from the second equation. Finally, we have

$$\psi'' - 2\tau\psi' + 4(e^{\psi} - 1) = \mathrm{const}/\tau^2. \tag{10}$$

For solutions of Eq. (10) that are finite for $\tau = 0$, we must set const = 0. Moreover, the condition

$$\varphi \to 0 \text{ for } |\tau| \to \infty, \tag{11}$$

must be satisfied, since $\varphi > 0$ for any $\tau$. Self-similar solutions of this kind describe a developing current filament in which the electron density is far higher than in the uniform plasma.

Symmetric solutions of an equation analogous to (10) were investigated in [3-5], in which the authors started from a power-law dependence of the power of the nonlinear sources on the temperature. The temperature dependence of the ionization frequency had the same form in [2]. The difference between the two approaches has a formal character in the given case, since with $C \gg 1$ and a small variation of T, the difference between the exponential function and the power-law function $\nu_i = \nu_{i0}(T/T_0)^C$ is insignificant. According to these papers, the condition (11) defines a problem of finding an eigenfunction; such a function exists and is unique, too.

Since (10) contains no parameters, the values of $\psi(0)$ and $\Delta\tau$ (the width of the filament) must have the order of unity. For larger $\tau$, the functions $\psi$ and $\phi$ have the asymptotic forms $\psi \cong -2 \ln \tau$ and $\phi \cong 1/\tau^2$. This means that the gas temperature and electron density far from the filament do not depend on time:

$$T \simeq T_0\left(-\frac{2}{C}\right)\ln\frac{x}{2\sqrt{\frac{D_a}{\nu_{i0}}}}, \quad n \simeq n_0\frac{4D_a}{x^2\nu_n}.$$

The total current in the filament grows with time as $(t_0 - t)^{-1/2}$. In the general case of $\kappa \neq 0$ and $k \neq 0$, the order of the system (8), (9) cannot be reduced. It proved possible to clarify the character of the individual solutions and make qualitative estimates, however.

The monotonicity of the behavior of the functions $\phi$ and $\psi$ can be judged from their behavior near the "equilibrium position" $\phi_0 = 1$, $\psi_0 = \ln(1 + k)$. Just as happened in [4], we find that for a small deiation of the function $\psi$ ($\psi_1 = \psi - \psi_0$) satisfying the linearized system (8), (9), there exists a unique solution not growing exponentially as $|\tau| \to \infty$:

$$\psi_1 \sim \tau^2 - (1 + k\varkappa)/2(1 + k). \tag{12}$$

This gives reason to assume that, although the order of the system is higher for $\kappa \neq 0$, the eigenfunction satisfying the condition (11) remains unique, and all the new solutions appearing with the increase in order have a nonphysical asymptotic form $\psi' \sim \exp(\tau^2/\kappa)$ as $|\tau| \to \infty$. The solution (12) approximately describes the behavior of the eigenfunction near the maximum and enables one to estimate the width of the filament. Taking $\psi_1 = 0$ in (12), we find

$$\Delta\tau = \left[\frac{1 + k\varkappa}{2(1 + k)}\right]^{1/2}.$$

The conditions of a concrete system determine the various contributions of transfer processes to the formation of the filament. Heat conduction should be neglected for $k\kappa \ll 1$. Then $\Delta\tau \cong (1 + k)^{-1/2}$ and $\psi(0) \cong 1 + \ln(1 + k)$. Inside the filament, if $k \gg 1$, ionization is balanced by recombination, while the size is determined by diffusion. For $k\kappa \gg 1$ and $k \gg 1$, the value of $\Delta\tau$ is determined by heat conduction and has the order $\sqrt{\kappa}$. In this case the electron density and the gas temperature are connected locally $\phi = (1/k)e^{\psi}$, while the function $\psi$ satisfies an equation analogous to (10). Under these conditions of a gas discharge $\kappa \ll 1$ and, as a rule, $k\kappa \ll 1$.

Let us estimate the time of explosive development of instability. For an initial filament width $\Lambda$ equal to the distance between the walls (for determinacy, $\Lambda = 1$ cm), a diffusion coefficient $D_a = 10^3$ cm³/sec, and $k = 10$, we obtain $t_0 \cong \Lambda^2/4D_a k = 2.5 \cdot 10^{-5}$ sec.

Let us examine the efficiency of negative feedback between the density of the electrons and their temperature, accomplished through the ballast resistance in the discharge circuit. Let us assume that the electrodes are infinitely finely sectioned, while each section is connected to the current source through a separate ballast resistance. In this case the feedback is local and is described quantitatively by the equation for the external circuit,

$$\sigma E R_b + EL = \varepsilon, \tag{13}$$

where $R_b$ is the value of the specific ballast resistance, calculated per unit area. It is sufficient to consider small variations of the electric field, which are linearly connected with the electron density, in accordance with (13):

$$\frac{\Delta E}{E_0} = -\frac{\sigma(n_0) R_b}{L} \frac{n - n_0}{n_0} = -\alpha(v - 1).$$

Solutions of the system (1), (2) will not have an explosive character if the ionization frequency ceases to depend sharply on the gas temperature alone. In accordance with (3), (6), and (7), this happens for $\nu_n(t_0 - t) \cong \alpha$. At this time, at the maximum of nonuniformity, $n = n_0 + n_0/\alpha C$.

If we assume that a filament appears when $n = 2n_0$ ($n_0$ is the electron density of the uniform background), then the value $\alpha_{cr} \approx 1/C$ (the ratio of the specific ballast resistance to the specific resistance of the discharge) proves to be critical for the appearance of instability with an explosive character; it is determined in the limit of infinitely sectioned electrodes. In an actual situation, in general, there is an integral connection between the field and the discharge current [6], so that higher values of $\alpha_{cr}$ should be expected.

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